$$\frac{dy}{d\phi}(R+r)\cos\phi-(R+r)\cos\frac{R+r}{r}\phi.$$

But $\dfrac{ds^2}{d\phi^2}=\dfrac{dx^2}{d\phi}+\dfrac{dy^2}{d\phi^2}=4(R+r)^2\sin^2\dfrac{R}{2r}\phi.$

$\therefore$ Length of curve between cusp and cusp

$$=2\int_0^{2+r/R}(R+r)\sin\frac{R}{2r}\phi=\frac{8r}{R}(R+r).$$

In the case of the hypocycloid we have in its equation only to put $-r$ in lieu of r, and by proceeding in the same way we obtain the length of the curve

$$=\frac{8r}{R}(R-r).$$

AVERAGE AND PROBABILITY.

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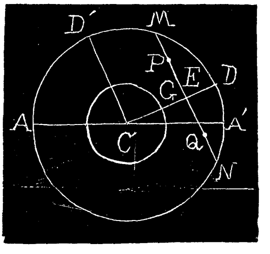
86. Proposed by L. C. WALKER, Assistant in Mathematics in Leland Stanford, Jr., University. Palo Alto, Cal.

Two points are taken at random in a circular annulus formed by two concentric circles. Find the chance that the straight line joining the points will not cut the inner variable circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let P, Q be the two random points, MN the chord through P, Q



Let $AC=r$, $CE=w$, $MQ=x$, $PQ=y$, $\angle ACD'=\theta$. CG, the radius of the variable circle$=u$, p=required chance.

An element of the circle at Q is $dwdx$, at P $yd\theta dy$. The limits of u are 0 and r; of w, r and u; of x, $2\sqrt{(r^2-w^2)}$ and 0; of y, 0 and x and doubled; of θ, 0 and 2π.

$$\therefore\ p=\frac{\dfrac{2}{\pi^2r^4}\displaystyle\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}\int_0^x\int_0^{2\pi}dudwdxydyd\theta}{\displaystyle\int_0^r du}$$

$$=\frac{4}{\pi r^5}\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}\int_0^x dudwdxydy$$

$$=\frac{2}{\pi r^5}\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}x^2dudwdx$$

$$=\frac{16}{3\pi r^5}\int_0^r\int_u^r(r^2-w^2)^{\frac{3}{2}}\,dudw$$

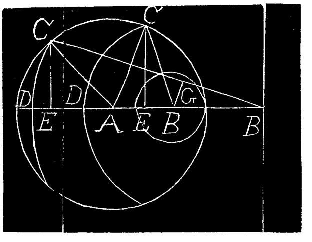
$$=\frac{1}{3\pi r^5}\int_0^r[3\pi r^4-10r^2u\sqrt{(r^2-u^2)}+4u^3\sqrt{(r^2-u^2)}-6r^4\sin^{-1}\frac{u}{r}]du$$

$$=\frac{16}{15\pi}=.3395.$$

87. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Find the mean distance of a random point in a sphere from a point, (1) within, (2) without the sphere.

Solution by the PROPOSER.



Let B be the given point. $AB=c$, $AC=a$, $AD=\pm x$.

Then $DB=c-x=BC$, $CE=\sqrt{(a^2-AE^2)}$
$=\sqrt{[(c-x)^2-(c-AE)^2]}$.

$\therefore\ AE=\dfrac{(a^2+2cx-x^2)}{2c}$,

$$DE=AE-x=\frac{(a^2-x^2)}{2c}.$$

Area of film $CDM=2\pi DB.DE=2\pi(c-x)(a^2-x^2)/2c$.

Also let $BG=y$. Then the area of the surface of this sphere is $4\pi y^2$.

$$(1).\ \text{Distance } D=\frac{\int_{-a}^{2c-a}2\pi DB^2.DEdx+\int_0^{a-c}4\pi y^3dy}{\frac{4}{3}\pi a^3}$$

$$\therefore\ D=\frac{3}{4a^3c}\int_{-a}^{2c-a}(c-x)^2(a^2-x^2)dx+\frac{3}{a^3}\int_0^{a-c}y^3dy.$$

$$\therefore\ D=\tfrac{3}{4}a+\frac{c^2}{2a}-\frac{c^4}{20a^3}.$$

$$(2).\ D_1=\frac{\int_{-a}^{a}2\pi DB^2.DEdx}{\frac{4}{3}\pi a^3}=\frac{3}{4a^3c}\int_{-a}^{a}(c-x)^2(a^2-x^2)dx.$$

$$\therefore\ D_1=c+(a^2/5c).$$